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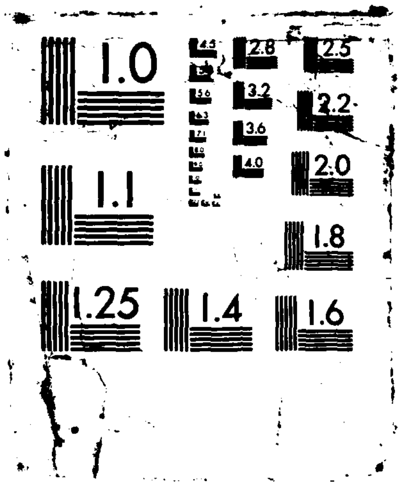
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July 1987

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Abstract

Solutions of stochastic differential equations having differentials of bounded variation processes on the right hand side can be defined by means of Lebesgue-Stieltjes integrals or by continuous extension of Stieltjes integrals. Both solutions are compared here and formulas that extend the Wong-Zakai theorem are obtained.

Key words and phrases: stochastic equations, Stratonovich integration, Wong-Zakai Theorem.

§1. Introduction

Solutions of stochastic differential equations having differentials of bounded variation processes on the right hand side are usually defined by means of pathwise Lebesgue-Stieltjes integrals (see, for example, Meyer [2]). Partially motivated by stochastic control problems we have introduced a different idea of solution in Ferreyra [1]. With this notion, the change of variables formula for solutions of stochastic differential equations driven by certain semimartingales is not complicated by the jumps of the processes involved. In fact, our interpretation is an extension of the concept of solution of stochastic differential equations in Stratonovich sense, together with the usual change of variables formula. Moreover, robustness in the driving process is built into this definition of solution. Hence results complementary to those considering approximation of driving martingales such as in Protter [4] and Picard [3] follow.

In this paper we obtain formulas relating both definitions of solutions. These formulas are in the vein of the theorems of Wong-Zakai [5].

We introduce the necessary notation and definitions in Section 2 along with some material from Ferreyra [1]. We explain the goal of this paper once more at the end of Section 2. In Section 3 a simple example is presented. Finally, Section 4 gives several formulas of the Wong-Zakai type.

§2. Notation, prerequisites and hypotheses

Let $(\Omega,\mathcal{F},P)$ be a complete probability space together with an increasing family of sub σ-algebras $\mathcal{F}_t$, $0 \leq t \leq T < \infty$, such that $\mathcal{F}_0$ contains all P-null elements of $\mathcal{F}$, and $\mathcal{F}_t$, $0 \leq t \leq T$, is right-continuous, that is, $\mathcal{F}_t = \mathcal{F}_{t^+} = \bigcap_{t<s<T} \mathcal{F}_s$,

for all t.

Two processes $X(t)$, and $Y(t)$, $0 \leq t \leq T$, are identified if they are *indistinguishable*, i.e., for almost all $\omega \in \Omega$ the equality $X(t,\omega) = Y(t,\omega)$ holds for all t. A process with paths which are continuous on the right (respectively, left) and have limits on the left (respectively, right) will be called *corlol* (respectively, *collor*). Other authors use the French versions cadlag and caglad respectively. We assume that all martingales (but not all processes) are corlol. If $X(t)$, $0 \leq t \leq T$ is a corlol or a cadlag process, then $\Delta X(t)$ denotes the jump $X(t^+) - X(t^-)$ at t. Two σ-fields on $[0,T] \times \Omega$ are of importance to us. The *optional* σ-field Σ_0 which is generated by the family of all adapted corlol processes and the *predictable* σ-field Σ_p which is generated by the adapted collor processes. A process $X(t)$, $0 \leq t \leq T$, is said to be optional (respectively, predictable) if it is Σ_0-measurable (respectively, Σ_p-measurable). A process is said to be of *bounded variation* if it is adapted, corlol, and it has paths of bounded variation. A process of bounded variation $A(t)$, $0 \leq t \leq T$, is said to be of *integrable variation* if $E\int_{0^-}^{T}|dA(s)| < \infty$. It is assumed that $A(t) = 0$, $t < 0$. If $A(t)$, $0 \leq t \leq T$ is of integrable variation and $H(t)$, $0 \leq t \leq T$, is an optional process such that $E\int_0^\infty |H(s)|\,|dA(s)| < \infty$, then the *stochastic integral* $I(t) = \int_0^t H(s)dA(s)$ is well defined as a pathwise Lebesgue-Stieltjes integral (cf. Meyer [2], p.258). The process $I(t)$, $0 \leq t \leq T$, turns out to be adapted, continuous on the right and of integrable variation. An adapted process $M(t)$, $0 \leq t \leq T$, vanishing at zero is called a *local martingale* if there exist stopping times $T_n \uparrow T$ such that the stopped process $M^{T_n}(t) =: M(t \wedge T_n)$ are uniformly integrable martingales. An adapted process $Z(t)$ is a *semimartingale* if it admits a decomposition of the form $Z(t) = Z(0) + M(t) + A(t)$, where $M(t)$ is a local martingale vanishing at zero, and $A(t)$ is a process of bounded variation vanishing at zero. If $Z(t)$ is a semimartingale, $Z^c(t)$ will denote its *continuous part*. If $M(t)$ is a square integrable martingale we let $\langle M,M\rangle(t)$ denote the unique *increasing predictable process* such that $\langle M,M\rangle(0) = M^2(0)$ and $M^2(t) - \langle M,M\rangle(t)$ is a martingale. If $M(t)$ and $N(t)$ are both square integrable martingales, then $\langle M,N\rangle(t) = \frac{1}{2}(\langle M + N, M + N\rangle(t) - \langle M,M\rangle(t) - \langle N,N\rangle(t))$ is the unique *predictable process of integrable variation* such that $\langle M,N\rangle(0) = M(0)N(0)$ and $M(t)N(t) - \langle M,N\rangle(t)$ is a martingale. We are ready now to state Ito's rule for semimartingales.

Theorem (Meyer [2], p. 301): *Let* $Z(t)$ *be an* $\mathbb{R}^n$*-valued process such that each one of its components* $Z^i(t)$, $i = 1,\cdots,n$, *is a semimartingale. Let* $F \in C^2(\mathbb{R}^n)$. *Then* $F(Z(t))$ *is a semimartingale and*

(1)
$$F(Z(t)) = F(Z(0)) + \sum_{i=1}^{n} \int_{0^+}^{t} \frac{\partial F}{\partial x_i} (Z(s^-))dZ^i(s)$$
$$+ \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j} (Z(s^-))\, d\langle Z^{ic},Z^{jc}\rangle(s)$$
$$+ \sum_{0<s\leq t} (F(Z(s)) - F(Z(s^-)) - \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} (Z(s^-))\Delta Z(s)).$$

Next, we describe results obtained in Ferreyra [1]. The meaning of the expression

(2)
$$dX(t) = f(X(t))\circ du(t) + g(X(t))dt + \sum_{\nu=1}^{J} \sigma_\nu(X(t))\circ dW^\nu(t)$$

will be given below when we define what is meant by solving (2) with initial condition

(3) $$X(0) = X.$$

Let the process $(W^1(t),\cdots,W^J(t))$, $0 \leq t \leq T$, be a given J-dimensional Brownian motion. Assume that $u(t)$, $0 \leq t \leq T$, belongs to the set U of real valued, uniformly bounded, collor processes. Assume, without loss of generality, that all processes $v \in U$ satisfy $v(0) = 0$. Assume that $u(t) = 0$, for $t < 0$. The unknown process $X(t)$, $0 \leq t \leq T$, has values in $\mathbb{R}^n$. Furthermore, assume

(H1) $E|X|^p < \infty$, *for some* $p > 2$,

(H2) $f \in C^3(\mathbb{R}^n)$, $f_x \in C_b^2(\mathbb{R}^n)$,

(H3) $g \in C_b^1(\mathbb{R}^n)$, *and*

(H4) $\sigma_\nu \in C_b^2(\mathbb{R}^n)$, $\nu = 1,\cdots,J$.

Let $\mathfrak{L}$ denote the set of adapted real valued processes $v(t)$, $0 \leq t \leq T$, having Lipschitz paths with a uniform Lipschitz constant. If $u(t) = v(t) \in \mathfrak{L}$, then $X(t)$ is said to solve (2) if

(4) $$dX(t) = f(X(t))\frac{dv}{dt}(t)dt + g(X(t))dt + \sum_{\nu=1}^{J} \sigma_\nu(X(t)) \circ dW^\nu(t)$$

in Stratonovich sense. It is well known that for such $v(t) \in \mathfrak{L}$, the problem (3) - (4) has a unique solution. This concept of solution is extended to allow all $u \in U$ as follows.

Definition 1: An $\mathbb{R}^n$-valued process $X(t)$, $0 \leq t \leq T$, is said to be a *solution* of (2) - (3) if there exists a map $\Gamma : [0,T] \times U \times \Omega \to \mathbb{R}^n$ such that

(D1) for each $v \in U$, $\Gamma(t,v)$, $0 \leq t \leq T$, is collor, (D2) if $v \in \mathfrak{X}$, then the process $\Gamma(t,v)$ solves (4) in Stratonovich sense,

(D3) if $v \in U$ and $\{v_j\}$ is a uniformly bounded sequence of elements in U such that for every t, $0 \leq t \leq T$, $v_j(t) \to v(t)$, a.s., then for each t, $E|\Gamma(t,v_j) - \Gamma(t,v)|^2 \to 0$, as $j \to \infty$,

(D4) $\Gamma(t,u) = X(t)$, $0 \leq t \leq T$, and

(D5) for all $v \in U$, $\Gamma(0,v) = X$.

The extension from $\mathfrak{X}$ to U is aided by the following.

Lemma 1 (Ferreyra [1]): *Let* $u \in U$, *and let* $u_j(t) = j\int_{t-1/j}^{t} u(s)ds$. *Then* $\{u_j\}$ *is a uniformly bounded sequence of elements in* $\mathfrak{X}$ *such that for each* t, $0 \leq t \leq T$, $u_j(t) \to u(t)$, a.s. as $j \to \infty$.

Given $u \in U$, the problem (2) - (3) is shown to have a unique solution in Ferreyra [1]. We sketch here the proof of existence. Let $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be the flow of f, that is, the solution of $\frac{\partial F}{\partial s}(s,x) = f(F(s,x))$, $F(0,x) = x$. Let $\bar{g}(s,x) = \left[\frac{\partial F}{\partial x}(s,x)\right]^{-1} g(F(s,x))$, and $\bar{\sigma}_\nu(s,x) = \left[\frac{\partial F}{\partial x}(s,x)\right]^{-1} \sigma_\nu(F(s,x))$. Then Γ is defined by

(5) $$\Gamma(t,v) = F(v(t),Y(t)),$$

where $Y(t)$ is the process with continuous paths satisfying

(6) $$dY(t) = \bar{g}(v(t),Y(t))dt + \sum_{\nu=1}^{J} \bar{\sigma}_\nu(v(t),Y(t)) \circ dW^\nu(t),$$

(7) $$Y(0) = X.$$

The proof of (D2) follows easily by application of the rule for change of variables for Stratonovich integrals. The proof of (D3) is a little more complicated. Basically, it involves estimates for F, its partial derivatives of first order, and estimates for $Y(t)$. The reader is referred to Ferreyra [1] for more details.

Finally, we restate the goal of this paper. We intend to find an expression for $X(t)$, the solution of (2) - (3), in terms of integrals of Lebesgue-Stieltjes and Itô type such as those appearing in (1).

§3. A simple example

To clarify the relation between (1) and (2) we consider the following deterministic example. Let $T = 3$, $u^- = \mathbf{1}_{(1,3]} + \mathbf{1}_{(2,3]}$, and $u^+ = \mathbf{1}_{[1,3]} + \mathbf{1}_{[2,3]}$, where for $A \subset [0,3]$, $\mathbf{1}_A$ denotes the characteristic function of the set A. Consider the 2-dimensional system of the type of (2) - (3)

$$dX^1(t) = 1 \circ du^-(t), \qquad X^1(0) = 0,$$

$$dX^2(t) = \psi(X^1(t)) \circ du^-(t), \qquad X^2(0) = 0,$$

where $\psi(x) = 3x^2$.

The solution of this system is computed using Definition 1 as follows. Let $u_j(t) = j(t-1 + 1/j)\mathbf{1}_{(1-1/j,1]}(t) + j(t-2 + 1/j)\mathbf{1}_{(2-1/j,2]}(t) + u^-(t)$. Then solve by standard methods

$$dX_j^1(t) = \frac{du_j}{dt}(t)\,dt, \qquad X_j^1(0) = 0,$$

$$dX_j^2(t) = \psi(X_j^1(t))\frac{du_j}{dt}(t)dt, \quad X_j^2(0) = 0.$$

Finally, take the limit as $j \to \infty$. Hence

$$X_j^1(t) = u_j(t), \quad X_j^2(t) = \int_0^t \psi(u_j(s))du_j(s) = \left[u_j(t)\right]^3,$$

$$X^1(t) = u(t), \quad X^2(t) = \left[u^-(t)\right]^3.$$

On the other hand the 2-dimensional system of the type of (1) - (3)

$$dY^1(t) = du^+(t), \qquad Y^1(0) = 0$$

$$dY^2(t) = \psi(Y^1(t^-))du^+(t), \quad Y^2(0) = 0$$

is solved by means of Lebesgue-Stieltjes integrals as

$$Y^1(t) = u^+(t),$$

$$Y^2(t) = \int_0^t \psi(u^+(s^-))du^+(s) = \int_0^t 3\left[u^+(s^-)\right]^2 du^+(s).$$

Thus $X^2(t)$ jumps 1 unit at $t = 1$ and 7 units at $t = 2$, while $Y^2(t)$ jumps 0 units at $t = 1$ and 3 units at $t = 2$. The difference between $X^2(t)$ and $Y^2(t)$ can be deduced from (1). In fact,

$$X^2(t) - Y^2(t) = \sum_{0<s\leq t}\left\{\left[u^+(s)\right]^3 - \left[u^+(s^-)\right]^3 - 3\left[u^+(s^-)\right]^2\Delta u^+(s)\right\} - \Delta X^2(t)$$

$$= \sum_{0<s<t}\left\{X^2(s^+) - X^2(s) - 3\left[X^2(s)\right]^2\Delta u^-(s)\right\} -3[X^2(t)]^2\Delta u^-(t).$$

We will prove below several generalizations of this formula.

§4. Stochastic integrals and stochastic differential equations

In the previous section we found a formula relating $X^2(t) = \lim_{j\to\infty} \int_0^t \psi(u_j(s))du_j(s)$ and the Lebesgue-Stieltjes integral $Y^2(t) = \int_0^t \psi(u^+(s^-))du^+(s)$. More general situations are treated here.

We assume throughout this section that $u(t)$, $0 \leq t \leq T$, is a fixed proces in U such that it has paths of bounded variation. Hence the process $u(t^+)$, $0 \leq t < T$, is well defined and it is of bounded variation. Let $u_j(t)$, $j = 1,2,\cdots$, $0 \leq t \leq T$, be a sequence of processes in $\mathfrak{X}$ approximating $u(t)$ in the sense of Lemma 1, that is, for each $0 \leq t \leq T$, $u_j(t)$ converges to $u(t)$, a.s. as $j \to \infty$.

Theorem 1: *Let* $\varphi = \varphi(t,x)$ *be a real valued function in* $C^1([0,T] \times \mathbb{R})$. *Then, for each* $0 \leq t \leq T$,

$$\lim_{j\to\infty} \int_0^t \varphi(s,u_j(s))du_j(s) = \psi(0,u(0^+)) + \int_{0^+}^t \varphi(s,u(s))du(s^+)$$

$$+ \sum_{0<s<t} \left[\psi(s,u(s^+)) - \psi(s,u(s)) - \varphi(s,u(s))\Delta u(s)\right] - \varphi(t,u(t))\Delta u(t),$$

where $\psi(t,x) = \int_0^x \varphi(t,\xi)d\xi$.

Proof: Let ψ be defined as above. Then, by the calculus for Riemann-Stieltjes integrals

$$\psi(t,u_j(t)) = \int_0^t \varphi(s,u_j(s))du_j(s) + \int_0^t \frac{\partial\psi}{\partial t}(s,u_j(s))ds.$$

Since for each t, $0 \leq t \leq T$, $u_j(t) \to u(t)$, a.s. as $j \to \infty$, then our hypotheses on φ and the Dominated Convergence Theorem imply

$$\psi(t,u(t)) = \lim_{j\to\infty} \int_0^t \varphi(s,u_j(s))du_j(s) + \int_0^t \frac{\partial\psi}{\partial t}(s,u(s))ds. \tag{8}$$

But (1) with $n = 2$, $Z^1(t) = t$, $Z^2(t) = u(t^+)$ imply

$$\psi(t,u(t^+)) = \psi(0,u(0^+) + \int_{0^+}^t \varphi(s,u(s))du(s^+) + \int_0^t \frac{\partial\psi}{\partial t}(s,u(s))ds$$
$$+ \sum_{0<s\leq t} \left[\psi(s,u(s^+)) - \psi(s,u(s)) - \varphi(s,u(s))\Delta u(s)\right]. \tag{9}$$

Comparison of (8) and (9) prove the desired formula.

The above theorem is further generalized as follows. Let $\alpha,\beta_\nu \in C([0,T])$, $\nu = 1,\cdots,J$, and define

$$S_j(t) = \int_0^t \alpha(s)du_j(s) + \sum_{\nu=1}^{J} \int_0^t \beta_\nu(s) \circ dW^\nu(s).$$

Here α and β_ν are deterministic functions, the first integral is pathwise of Riemann-Stieltjes type and the integrals with respect to W^ν are of Stratonovich (equal to Itô in this case) type. Let $\varphi = \varphi(t,x) \in C^2([0,T] \times \mathbb{R})$ and consider the stochastic integral

$$\int_0^t \varphi(s,S_j(s)) \circ dS_j(s) =: \int_0^t \varphi(s,S_j(s))\alpha(s)du_j(s) + \sum_{\nu=1}^{J} \int_0^t \varphi(s,S_j(s))\beta_\nu(s) \circ dW^\nu(s).$$

Finally, define the (collor) process

$$S(t) = \int_0^t \alpha(s)du(s) + \sum_{\nu=1}^{J} \int_0^t \beta_\nu(s)dW^\nu(s),$$

where the first integral is of Riemann-Stieltjes type and the other J integrals are of Itô (equal to Stratonovich in this case) type.

Theorem 2: *Let* $\psi(t,x) = \int_0^x \varphi(t,\xi)d\xi$. *Then, for each* $0 \leq t \leq T$, *the following relation holds.*

$$\lim_{j\to\infty} \int_0^t \varphi(s,S_j(s)) \circ dS_j(s) = \psi(0,S(0^+)) + \int_{0^+}^t \varphi(s,S(s))dS(s^+) + \frac{1}{2}\sum_{\nu=1}^{J}\int_0^t \frac{\partial\varphi}{\partial x}(s,S(s))\beta_\nu^2(s)ds$$
$$+ \sum_{0<s<t} \left[\psi(s,S(s^+)) - \psi(s,S(s)) - \varphi(s,S(s))\Delta S(s)\right] - \varphi(t,S(t))\Delta S(t).$$

Proof: By the calculus for Stratonovich integrals

$$\psi(t,S_j(t)) = \int_0^t \varphi(s,S_j(s)) \circ dS_j(s) + \int_0^t \frac{\partial\psi}{\partial t}(s,S_j(s))ds.$$

Theorem 1 implies that

$$\lim_{j\to\infty} \int_0^t \alpha(s)du_j(s) = \alpha(0)u(0^+) + \int_{0^+}^t \alpha(s)du(s^+) - \alpha(t)\Delta u(t) = \int_0^t \alpha(s)du(s).$$

Then, for each $0 \leq t \leq T$, $S_j(t) \to S(t)$, a.s. as $j \to \infty$. Hence the Dominated Convergence Theorem implies

(10) $$\psi(t,S(t)) = \lim_{j\to\infty} \int_0^t \varphi(s,S_j(s)) \circ dS_j(s) + \int_0^t \frac{\partial\psi}{\partial t}(s,S(s))ds.$$

On the other hand, (1) with $n = 2$ and $Z^1(t) = t$, $Z^2(t) = S(t^+)$ give

(11) $$\psi(t,S(t^+)) = \psi(0,S(0^+)) + \int_{0^+}^t \varphi(s,S(s))dS(s^+) + \frac{1}{2}\sum_{\nu=1}^{J}\int_0^t \frac{\partial\varphi}{\partial x}(s,S(s))\beta_\nu^2(s)ds$$
$$+ \int_0^t \frac{\partial\psi}{\partial t}(s,S(s))ds + \sum_{0<s\leq t}\Big[\psi(s,S(s^+)) - \psi(s,S(s)) - \varphi(s,S(s))\Delta S(s)\Big].$$

Putting (10) and (11) together we obtain our result.

Theorem 3: *Let* $X(t)$, $0 \leq t \leq T$, *be the solution of* (2) - (3) *in the sense of Definition* 1. *Then*

(12) $$X(t) = X + \int_{0^+}^t f(X(s))du(s^+) + \int_0^t g(X(s))ds + \sum_{\nu=1}^{J}\int_0^t \sigma_\nu(X(s))dW^\nu(s)$$
$$+ \frac{1}{2}\sum_{i=1}^{n}\sum_{\nu=1}^{J}\int_0^t \frac{\partial\sigma_\nu}{\partial x_i}(X(s))\sigma_\nu^i(X(s))ds + \sum_{0\leq s<t}\Delta X(s) - \sum_{0<s\leq t} f(X(s))\Delta u(s).$$

Proof: The solution of (2) - (3) can be expressed, according to (5)- (7), as

(13) $$X(t) = F(u(t), Y(t)),$$

where F is the flow of f and $Y(t)$, $0 \leq t \leq T$, is the continuous semimartingale solution of $Y(0) = X$, $dY(t) = \bar{g}(u(t),Y(t))dt +$ $+ \sum_{\nu=1}^{J} \bar{\sigma}_\nu(u(t),Y(t))\circ dW^\nu(t)$. As indicated at the beginning of this section, the process $u(t^+)$, $0 \leq t < T$, is of bounded variation. Then, it follows from (1) that

$$X(t^+) = F(u(0^+),Y(0)) + \int_{0^+}^t \frac{\partial F}{\partial s}(u(s),Y(s))du(s^+) + \sum_{i=1}^n \int_0^t \frac{\partial F}{\partial x_i}(u(s),Y(s))dY^i(s)$$

$$+ \frac{1}{2} \sum_{i,j=1}^n \int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}(u(s),Y(s))d\langle Y^i,Y^j\rangle(s)$$

$$+ \sum_{0<s\leq t} \Big[F(u(s^+),Y(s)) - F(u(s),Y(s)) - \frac{\partial F}{\partial s}(u(s),Y(s))\Delta u(s)\Big].$$

The following equalities are used to replace the various terms in the above equation. It is easy to see that

$$F(u(s^+),Y(s)) - F(u(s),Y(s)) = \Delta X(s),$$

$$F(u(0^+),Y(0)) = X + \Delta X(0),$$

$$X(t^+) = X(t) + \Delta X(t),$$

$$\frac{\partial F}{\partial s}(u(s),Y(s)) = f(F(u(s),Y(s))) = f(X(s)),$$

$$\frac{\partial F}{\partial x}(u(s),Y(s))\bar{g}(u(s),Y(s)) = g(X(s)),$$

$$\frac{\partial F}{\partial x}(u(s),Y(s))\bar{\sigma}_\nu(u(s),Y(s)) = \sigma_\nu(X(s)),$$

and

$$\sum_{i=1}^n \frac{\partial F}{\partial x_i}(u(s),Y(s)) \sum_{\nu=1}^J \sum_{j=1}^n \frac{\partial \bar{\sigma}_\nu^i}{\partial x_j}(u(s),Y(s))\bar{\sigma}_\nu(u(s),Y(s)) =$$

$$= -\sum_{i,j=1}^n \frac{\partial^2 F}{\partial x_i \partial x_j}(u(s),X(s))\ d\langle Y^i,Y^j\rangle(s) + \sum_{\nu=1}^J \sum_{i=1}^n \frac{\partial \sigma_\nu}{\partial x_i}(X(s))\sigma_\nu^i(X(s)).$$

In the last equality we used the identity obtained from differentiation of $F_x(s,x)F_x(s,x)^{-1} = I$ with respect to x. The proof of (12) is then concluded.

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